

Handlebody Splittings of Compact 3-Manifolds with Boundary

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ABSTRACT

The purpose of this paper is to relate several generalizations of the notion of the Heegaard splitting of a closed 3-manifold to compact, orientable 3-manifolds with nonempty boundary.

Key words: 3-manifolds with boundary, Heegaard splittings.

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1. Introduction

Throughout this paper we work in the piecewise-linear category, consisting of simplicial complexes and piecewise-linear maps.

We call a compact, connected, orientable 3-manifold M with nonempty boundary ∂M a *bordered* 3-manifold. A bordered 3-manifold H is said to be a *handlebody of genus g* iff H is the disk-sum (i.e., the boundary connected-sum) of g copies of the solid-torus $D^2 \times S^1$ (see Gross [3], Swarup [16], etc.). A handlebody of genus g is characterized as a regular neighborhood $N(P; \mathbb{R}^3)$ of a connected 1-polyhedron P with Euler characteristic $\chi(P) = 1 - g$ in the 3-dimensional Euclidean space \mathbb{R}^3 and as an irreducible bordered 3-manifold M with connected boundary whose fundamental group $\pi_1(M)$ is a free group of rank g (see Ochiai [10]).

It is well-known that a closed (i.e., compact, without boundary), connected, orientable 3-manifold M is decomposed into two homeomorphic handlebodies; that is,

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Proposition 1.1 (Heegaard Splittings; see Seifert-Threlfall [14], etc.).

- (i) For every closed, connected, orientable 3-manifold M , there exist handlebodies H_1 and H_2 in M such that
 - (a) $H_1 \cong H_2$, that is, $\text{genus}(H_1) = \text{genus}(H_2) = g$,
 - (b) $M = H_1 \cup H_2$, and
 - (c) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = \partial H_1 = \partial H_2 = F$, the Heegaard surface.
- (ii) For every bordered 3-manifold M , there exist a handlebody H_1 and a disjoint union of 2-handles (i.e., 3-balls) $H_2 = h_1 \cup \cdots \cup h_s$ such that
 - (a) $\text{genus}(H_1) = g$,
 - (b) $M = H_1 \cup H_2$, and
 - (c) each h_i attached to H_1 at $\partial H_1 = F$, the Heegaard surface.

We call such a $(M; H_1, H_2; F)$ a *Heegaard splitting* (or *H-splitting*) for M of genus g , and call the minimum genus of such splittings for M the *Heegaard genus* (or *H-genus*) of M and denote it by $\text{Hg}(M)$.

For an H-splitting for a closed orientable 3-manifold, Haken [4] proved the following fundamental theorem (see Hempel [5], Jaco [6], and also Ochiai [11]):

Proposition 1.2 (Haken [4]). *If a closed orientable 3-manifold M with a given Heegaard splitting $(M; H_1, H_2; F)$ contains an essential 2-sphere, then M contains a 2-sphere which meets F in a single circle.*

Since H_2 of an H-splitting for a bordered 3-manifold M is a disjoint union of 3-balls and so $\partial H_2 \neq F$, a Haken type theorem cannot be formulated for a H-splitting for M . Casson-Gordon [1] introduced the concept of compression bodies as a generalization of handlebodies, and for a bordered 3-manifold they defined a new Heegaard splitting using compression bodies, and formulated and proved a generalization of Haken's theorem.

On the other hand, in 1970 Downing [2] proved that every bordered 3-manifold can be decomposed into two homeomorphic handlebodies, and Roeling[13] discussed on these decompositions for bordered 3-manifolds with connected boundary. The purpose of the paper is to report the Downing's results [2] and Roeling's results [13] in slightly modified and generalized forms, and formulate a Haken type theorem for these decompositions in the way of Casson-Gordon [1].

2. Handlebody-splittings for bordered 3-manifolds

For a bordered 3-manifold M , let $\partial M = B_1 \cup B_2 \cup \cdots \cup B_m$, here B_i is a connected component for $i = 1, 2, \dots, m$, and let $g_i = \text{genus}(B_i)$.

Theorem 2.1 (Downing [2]). *For every bordered 3-manifold M , there exist handlebodies H_1 and H_2 in M which satisfy the following:*

- (i) $H_1 \cong H_2$, that is, $\text{genus}(H_1) = \text{genus}(H_2) = g$,
- (ii) $M = H_1 \cup H_2$,
- (iii) $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = F_0$ is a connected surface, the splitting-surface,
- (iv) $H_j \cap B_i = \partial H_j \cap B_i = K_{ji}$ is a disk with g_i holes, and $K_{1i} \cong K_{2i}$ ($j = 1, 2$, $i = 1, 2, \dots, m$),
- (v) the homomorphism induced from the inclusion

$$\iota : \pi_1(K_{ji}; x_i) \rightarrow \pi_1(H_j; x_i), \quad x_i \in \partial K_{ji} \quad (j = 1, 2, i = 1, 2, \dots, m)$$

is injective.

We call such a $(M; H_1, H_2; F_0)$ a *Downing splitting* (or *D-splitting*) for M of genus g , and call the minimum genus of such splittings for M the *Downing genus* (or *D-genus*) of M and denote it by $\text{Dg}(M)$. By the way, Roeling [13] has pointed out that $\pi_1(K_{ji}; x_i)$ in Theorem 2.1 (v) injects not only into $\pi_1(H_j; x_i)$ but also onto a free factor of $\pi_1(H_j; x_i)$, when the boundary ∂M is connected. In fact, it holds the following:

Theorem 2.2. *For every bordered 3-manifold M , there exists a D-splitting $(M; H_1, H_2; F_0)$ which satisfies the following:*

- (v) the homomorphism induced from the inclusion

$$\iota : \pi_1(K_{ji}; x_i) \rightarrow \pi_1(H_j; x_i), \quad x_i \in \partial K_{1i} = \partial K_{2i} \quad (j = 1, 2, i = 1, 2, \dots, m)$$

is injective, and every image $\iota\pi_1(K_{ji}; x_i)$ is a free factor of the free group $\pi_1(H_j; x_i)$ of rank g ,

- (vi) there exists a tree T in F_0 connecting x_1, x_2, \dots, x_m such that the homomorphism induced from inclusion

$$\iota : \pi_1(K_{j1} \cup \dots \cup K_{jm} \cup T; x) \rightarrow \pi_1(H_j; x), \quad x \in T \quad (j = 1, 2)$$

is injective, and the image is a free factor of $\pi_1(H_j; x)$.

By Zieschang [18, §3, Satz 2 and Korollar], the above conditions (v) and (vi) are equivalent to the following geometric condition:

Theorem 2.3. *For every bordered 3-manifold M , there exists a D-splitting $(M; H_1, H_2; F_0)$ which satisfies the following (see figure 1):*

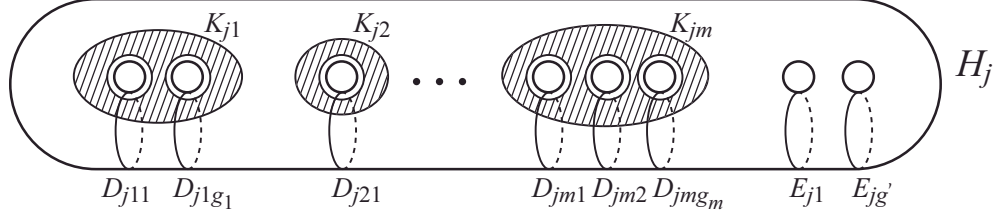


Figure 1

(vi*) there exist systems of meridian-disks $\mathcal{D}_{ji} = \{D_{ji1}, \dots, D_{jig_i}\}$ ($j = 1, 2, i = 1, 2, \dots, m$) and $\mathcal{E}_j = \{E_{j1}, \dots, E_{jg'}\}$, where $g' = g - (g_1 + \dots + g_m)$ of H_j satisfying the following:

- (a) $\mathcal{D}_{j1} \cup \dots \cup \mathcal{D}_{jm} \cup \mathcal{E}_j$ forms a complete system of meridian-disks of H_j ,
- (b) $D_{jik} \cap (K_{j1} \cup \dots \cup K_{jm}) = \partial D_{jik} \cap K_{ji}$ consists of a single simple arc, and $E_{jk} \cap K_{ji} = \emptyset$ ($j = 1, 2, i = 1, 2, \dots, m, k = 1, 2, \dots, g_i$), and
- (c) $\text{Cl}(K_{ji} - N(D_{ji1} \cup \dots \cup D_{jig_i}; H_j))$ is a disk ($j = 1, 2$).

According to Roeling [13], we call a D-splitting for M satisfying the conditions (v) and (vi) in Theorem 2.2 or the condition (vi*) in Theorem 2.3 a *special Downing splitting* (or *SD-splitting*) for M of genus g , and call the minimum genus of such splittings for M the *special Downing genus* (or *SD-genus*) of M and denote it by $\text{SDg}(M)$.

It will be noticed that for a closed, connected, orientable 3-manifold, the three splittings, an H-splitting, a D-splitting and an SD-splitting, are considered as the same one.

In order to prove Theorems 2.1 and 2.2, we need a lemma which is a generalization of Lemma 1 of Downing [2]. In proving the lemma, the notation and definitions of Downing [2] will be helpful. If g is a nonnegative integer, let $Y(g)$ be the set of all points (x, y) in the plane \mathbb{R}^2 which satisfy

$$x \in \{0, 1, \dots, g\} \quad \text{and} \quad -1 \leq y \leq 1$$

or

$$0 \leq x \leq g \quad \text{and} \quad |y| = 1.$$

We put

$$\begin{aligned} X(g) &= \{(x, y) \in Y(g) \mid y \geq 0\}, \\ \partial X(g) &= \{(x, 0) \in X(g)\}, \\ Z(g) &= \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq g, 0 \leq y \leq 1\}. \end{aligned}$$

Let H be a handlebody with X a copy of $X(g)$ embedded as a PL subspace of H . X is said to be *proper* in H if $X \cap \partial H = \partial X$, and X is said to be *unknotted* if X is proper

in H and the embedding of $X(g)$ can be extended to an embedding of $Z(g)$. Let $X_1 \cup \cdots \cup X_m$ be a copy of $X(g_1) \cup \cdots \cup X(g_m)$ properly embedded as a PL subspace of H . We say that $X_1 \cup \cdots \cup X_m$ is *unknotted* if the embedding of $X(g_1) \cup \cdots \cup X(g_m)$ can be extended to an embedding of $Z(g_1) \cup \cdots \cup Z(g_m)$.

Lemma 2.4 (Downing [2]). *Let M' be a closed, connected orientable 3-manifold, and $(M'; W_1, W_2; F)$ be an H -splitting for M' . Let S be a 1-dimensional spine of W_1 . We suppose that $Y_1 \cup \cdots \cup Y_m$ is a copy of $Y(g_1) \cup \cdots \cup Y(g_m)$ embedded in S . Then there exists an ambient isotopy $\{\eta_t\}$ of M' satisfying the following:*

$\eta_1(Y_1 \cup \cdots \cup Y_m) \cap W_j = X_{j1} \cup \cdots \cup X_{jm}$ is a copy of $X(g_1) \cup \cdots \cup X(g_m)$ which is proper and unknotted in W_j for $j = 1, 2$.

Proof. The case $m = 1$ is Lemma 1 of Downing [2], and the proof of the case $m \geq 2$, which is omitted here, is the same as that of the case $m = 1$. \square

Proof of Theorems 2.1 and 2.2. The proof of Theorems 2.1 and 2.2 is almost similar to that of Theorem 1 of Downing [2] except for obvious modifications, but for future reference, we record it here.

Let V_i be a handlebody of genus g_i ($i = 1, 2, \dots, m$). We sew V_i into the boundary component B_i of M to form a closed, connected, orientable 3-manifold $M' = M \cup V_1 \cup \cdots \cup V_m$. Let Y_i be a copy of $Y(g_i)$ which is embedded as a 1-dimensional spine of V_i and we triangulate M' so that $Y_1 \cup \cdots \cup Y_m$ is contained in the 1-skeleton S .

Let $W_1 = N(S; M')$, a regular neighborhood of S in M' , and let $W_2 = \text{Cl}(M' - W_1; M')$. Then these form an H -splitting $(M'; W_1, W_2; F)$ for M' , where $F = \partial W_1 = \partial W_2$. By Lemma 2.4, there exists an ambient isotopy $\{\eta_t\}$ of M' so that

$\eta_1(Y_1 \cup \cdots \cup Y_m) \cap W_j = X_{j1} \cup \cdots \cup X_{jm}$ is a copy of $X(g_1) \cup \cdots \cup X(g_m)$ which is proper and unknotted in W_j ($j = 1, 2$). (*)

We put

$$N = N(\eta_1(Y_1 \cup \cdots \cup Y_m); M'),$$

$$N_1 = N(X_{11} \cup \cdots \cup X_{1m}; W_1), \quad N_2 = N(X_{21} \cup \cdots \cup X_{2m}; W_2).$$

Then, $N = N_1 \cup N_2$, and $\text{Cl}(M' - N)$ is homeomorphic to M because $\{\eta_t\}$ is an ambient isotopy. From the unknotted condition (*),

$$H_1 = \text{Cl}(W_1 - N_1), \quad H_2 = \text{Cl}(W_2 - N_2)$$

are homeomorphic handlebodies decomposing $\text{Cl}(M' - N) = M$, and it is easily checked that this splitting satisfies the conditions (ii)–(vi) in Theorems 2.1 and 2.2, completing the proof. \square

Proof of Theorem 2.3. Let $\partial K_{ji} = J_{ji0} \cup J_{ji1} \cup J_{ji2} \cup \cdots \cup J_{jig_i}$, and we assume $x_i \in J_{ji0}$, $j = 1, 2$, $i = 1, 2, \dots, m$. Now, we can choose points $x_{jik} \in J_{jik}$ ($k = 1, 2, \dots, g_i$) and mutually disjoint simple proper arcs d_{jik} in K_{ji} which span x_i and x_{ik} so that $J_{ji1} \cup d_{ji1} \cup J_{ji2} \cup d_{ji2} \cup \cdots \cup J_{jig_i} \cup d_{jig_i}$ is a strong deformation retract of K_{ji} . Then, from the conditions (v) and (vi) in Theorem 2.2, the system of simple loops

$$\bigcup_{i=1}^m \{J_{ji1}, J_{ji2}, \dots, J_{jig_i}\}$$

satisfies the condition of Satz 2 in Zieschang [18, §3], and we have the required systems of meridian-disks $\mathcal{D}_{ji} = \{D_{ji1}, \dots, D_{jig_i}\}$ ($j = 1, 2$, $i = 1, 2, \dots, m$) and $\mathcal{E}_j = \{E_{j1}, \dots, E_{jg'}\}$, where $g' = g - (g_1 + \cdots + g_m)$ of H_j of the condition (vi*) in Theorem 2.3.

It is easy to check that the condition (vi*) implies the conditions (v) and (vi) in Theorem 2.2, and we complete the proof. \square

3. Genera of bordered 3-manifolds

From the definitions and the proofs of Theorems 2.1 and 2.2, we know:

Proposition 3.1. *For every bordered 3-manifold M , it holds the following:*

- (i) $\text{SDg}(M) \geq \text{Dg}(M)$.
- (ii) $\text{SDg}(M) \geq g_1 + \cdots + g_m = \text{the total genus of } \partial M$.

The following two theorems were proved by Roeling [13] when $m = 1$, and the proofs of the general case are almost the same as that of $m = 1$ under the condition (vi*).

Theorem 3.2 (Roeling [13, Theorem 1]). *If a bordered 3-manifold M has an SD-splitting $(M; H_1, H_2; F_0)$ of genus g , then M has an H-splitting of genus g .*

Proof. To make our notation consistent with Roeling [13], we will use the following notation in this proof and the proof of Theorem 3.4. If D is a disk, then $N(D)$ will denote a space homeomorphic to $D \times [-1, 1]$ where D corresponds to $D \times \{0\}$. We denote the 2-handles h_1, \dots, h_s by $N(D_1), \dots, N(D_s)$, where D_k is a disk for each k , $N(D_k) \cap N(D_h) = \emptyset$ if $k \neq h$, and $N(D_k) \cap H_1 = \partial D_k \cap \partial H_1$ corresponds to $\partial D_k \times [-1, 1]$ in $N(D_k)$.

From the condition (vi*), we can choose a complete system of meridian-disks

$$\mathcal{D}_{21} \cup \cdots \cup \mathcal{D}_{2m} \cup \mathcal{E}_2$$

of H_2 also satisfying the conditions (b) and (c). Then,

$$\text{Cl}\left(H_2 - \bigcup_{i=1}^m N(D_{2i1} \cup \cdots \cup D_{2ig_i}; H_2)\right)$$

is a handlebody of genus $g' = g - (g_1 + \cdots + g_m)$, and

$$X = \text{Cl}\left(H_2 - \bigcup_{i=1}^m N(D_{2i1} \cup \cdots \cup D_{2ig_i}) - N(E_{21} \cup \cdots \cup E_{2g'})\right)$$

is a ball. We choose a system of properly embedded disks $\mathcal{D}' = \{D'_1, \dots, D'_{m-1}\}$ of H_2 so that \mathcal{D}' is disjoint from the complete system of meridian-disks and

$$Y = \text{Cl}(X - N(D'_1 \cup \cdots \cup D'_{m-1}))$$

consists of $m - 1$ balls. Now,

$$H_1^* = H_1 \cup \bigcup_{i=1}^m (N(D_{2i1}) \cup \cdots \cup N(D_{2ig_i}))$$

is a handlebody of genus g by the condition (b). Now we conclude that

$$M \cong H_1^* \cup N(E_{21}) \cup \cdots \cup N(E_{2g'}) \cup N(D'_1) \cup \cdots \cup N(D'_{m-1}),$$

because each ball of Y meets this in a disk on their common boundary. \square

Corollary 3.3. *For every bordered 3-manifold M , it holds that*

$$\text{Hg}(M) \leq \text{SDg}(M).$$

Theorem 3.4 (Roeling [13, Theorem 2]). *If a bordered 3-manifold M has an H -splitting $(M; H_1, H_2; F)$ of genus g , then M has a D -splitting of genus g .*

Proof. If $m = 1$, the result has been proved in Roeling [12, Theorem 2], so we assume $m \geq 2$. We suppose that $H_2 = N(D_1) \cup \cdots \cup N(D_s)$. Then,

$$\text{Cl}\left(\partial H_1 - \bigcup_{k=1}^s N(D_k)\right)$$

consists of m connected orientable surfaces, say, S_1, \dots, S_m , where

$$S_i = B_i \cap \text{Cl}\left(\partial H_1 - \bigcup_{k=1}^s N(D_k)\right).$$

Here, B_i is a connected component of ∂M . Let $\alpha_k \cup \beta_k = \partial D_k \times \{-1\} \cup \partial D_k \times \{1\}$ be simple closed curves for $k = 1, \dots, s$. Then, 2-handles can be classified into two types as follows:

(type I) α_k and β_k are contained in some S_i , or

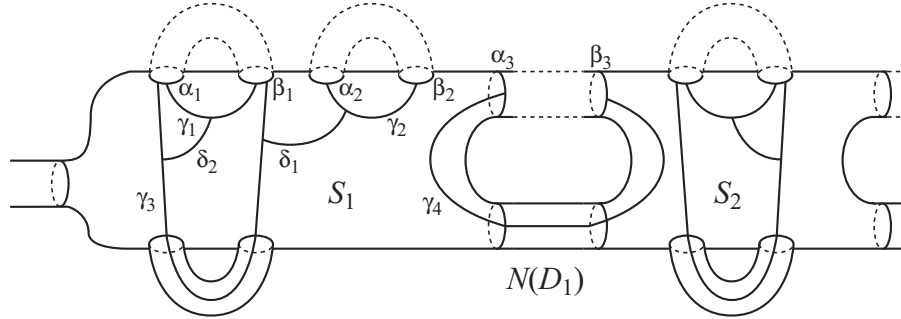


Figure 2

(type II) α_k is contained in a S_i and β_k is contained in a S_j with $i \neq j$.

We can choose $m - 1$ handles, say, $N(D_1) = D_1 \times [-1, 1], \dots, N(D_{m-1}) = D_{m-1} \times [-1, 1]$, so that

$$S = S_1 \cup \dots \cup S_m \cup \partial D_1 \times [-1, 1] \cup \dots \cup \partial D_{m-1} \times [-1, 1]$$

is connected, because $\partial H_1 = F$ is connected.

Now we choose simple, properly embedded, pairwise disjoint arcs $\gamma_m, \gamma_{m+1}, \dots, \gamma_s$ in S so that

- (i) each γ_k joins α_k to β_k ,
- (ii) if the 2-handle $N(D_k)$ is of type I, and α_k and β_k are contained in S_i , then $\gamma_k \subset S_i$, and if $N(D_k)$ is of type II, then γ_k crosses some of $\partial D_1, \dots, \partial D_{m-1}$ transversally, and
- (iii) $T' = \text{Cl}(S - \bigcup_{k=m}^s N(\gamma_k; S))$ is connected orientable surface of genus $g - s + m - 1$. See figure 2.

Now, we can check that

$$\chi(S) = 2 - 2g, \quad \chi(T') = 2 - 2g + (s - m + 1).$$

As indicated in figure 2, we choose properly embedded, pairwise disjoint simple arcs $\delta_1, \delta_2, \dots, \delta_{s-m}$ in T' so that

- (iv) each δ_k joins some γ_j to γ_r ($j \neq r$),
- (v) $T = \text{Cl}(T' - \bigcup_{k=1}^{s-m} N(\delta_k; T'))$ is connected.

Then, we know that

(vi) $S_i^* = Cl(S_i - \bigcup_{k=1}^s N(\gamma_k; S_i) - \bigcup_{k=1}^{s-m} N(\delta_k; S_i))$ is a disk with g_i holes for $i = 1, \dots, m$.

(vii) the inclusion induced homomorphism $\mu_* : \pi_1(S_i^*) \rightarrow \pi_1(S_i)$ is an injection.

[type I]: We assume that the inclusion induced homomorphism

$$\nu_i : \pi_1(S_i^*) \rightarrow \pi_1(H_1)$$

is not injective for some $i \in \{1, \dots, m\}$. Then, $\nu_i \mu_* : \pi_1(S_i^*) \rightarrow \pi_1(H_1)$ is not injective. We find by Dehn's lemma (see [5, 6]) a simple closed curve J in S_i^* that does not contract in S_i^* but bounds a disk E in H_1 . Cutting along E , either we separate M into manifolds M_1 and M_2 with H-splittings of genres $g(1) > 0$ and $g(2) > 0$ so that $g(1) + g(2) = g$, or we remove an 1-handle from M to get a manifold M_1 with an H-splitting of genus $g - 1$. Hence, by induction on g and the fact the theorem is trivial if $g = 1$, we are finished.

[type II]: We assume that the inclusion induced homomorphism

$$\nu_i : \pi_1(S_i^*) \rightarrow \pi_1(H_1)$$

is an injection for every $i = 1, \dots, m$. Then, $\nu_i \mu_* : \pi_1(S_i^*) \rightarrow \pi_1(H_1)$ is an injection. Let

$$H_2^* = \left(\bigcup_{k=1}^s N(D_k) \right) \cup \left(\bigcup_{k=m}^s N(\gamma_k; H_1) \right) \cup \left(\bigcup_{k=1}^{s-m} N(\delta_k; H_1) \right),$$

where

$$\left[\left(\bigcup_{k=m}^s N(\gamma_k; H_1) \right) \cup \left(\bigcup_{k=1}^{s-m} N(\delta_k; H_1) \right) \right] \cap S_i = \left(\bigcup_{k=m}^s N(\gamma_k; S_i) \right) \cup \left(\bigcup_{k=1}^{s-m} N(\delta_k; S_i) \right).$$

Let $H_1^* = Cl(H_1 - H_2^*)$. Then, H_1^* and H_2^* are handlebodies of genus g and $M = H_1^* \cup H_2^*$. Since the pair $(H_1^*, H_1^* \cap B_i)$ is homeomorphic to (H_1, S_i^*) , we have that $\pi_1(H_1^* \cap B_i)$ injects to $\pi_1(H_1^*)$. On the other hand, from our construction, we know that

$$H_2^* \cap B_i = \left(\bigcup_{k=1}^s (D_k \times \{-1, 1\}) \right) \cup \left(\bigcup_{k=m}^s N(\gamma_k; S_i) \right) \cup \left(\bigcup_{k=1}^{s-m} N(\delta_k; S_i) \right)$$

is a disk with g_i holes for every $i = 1, \dots, m$, and the inclusion induced homomorphism $\pi_1(H_2^* \cap B_i) \rightarrow \pi_1(H_2^*)$ is injective. Hence, M has a D-splitting of genus g . This completes the proof. \square

Corollary 3.5. *For every bordered 3-manifold M , it holds that*

$$Dg(M) \leq Hg(M) \leq SDg(M).$$

Closed 3-manifolds of H-genus 0 are characterized as the 3-dimensional sphere \mathbb{S}^3 . Corresponding to this fact, it holds the following:

Proposition 3.6. *Let M be a bordered 3-manifold with m boundary components.*

$$\begin{aligned} \text{SDg}(M) = 0 &\iff \text{Hg}(M) = 0 \\ &\iff M = \mathbb{S}^3 \text{ with } m \text{ holes} \\ &\iff M \text{ is the connected sum of } m \text{ copies of the 3-ball } D^3. \end{aligned}$$

Two H-splittings $(M; H_1, H_2; F)$ and $(M; H'_1, H'_2; F')$ for a 3-manifold M are said to be *equivalent*, if there exists a homeomorphism $\psi : M \rightarrow M$ with $\psi(F) = F'$. Let $(M; H_1, H_2; F)$ be an H-splitting for M of genus g , and let $(\mathbb{S}^3; U_1, U_2; T^2)$ be an H-splitting for the 3-sphere \mathbb{S}^3 of genus 1. Remove a 3-ball from M and a 3-ball from \mathbb{S}^3 , choosing these 3-balls so that they meet the respective Heegaard surfaces in disks. Then, if we use these 3-balls to form the connected sum $M \# \mathbb{S}^3 \cong M$ of M and \mathbb{S}^3 , we shall obtain a new H-splitting for M with Heegaard surface $F \# T^2$ of genus $g + 1$, and we denote this splitting by $(M; H_1, H_2; F) \# (\mathbb{S}^3; U_1, U_2; T^2)$. This process is called stabilizing; it may be iterated to obtain H-splittings $(M; H_1, H_2; F) \# n(\mathbb{S}^3; U_1, U_2; T^2)$ of any genus $g + n > g$. Two H-splittings $(M; H_1, H_2; F)$ and $(M; H'_1, H'_2; F')$ are said to be *stably equivalent*, if there exist integers n, n' with $h = g + n = g' + n'$ so that the stabilizations $(M; H_1, H_2; F) \# n(\mathbb{S}^3; U_1, U_2; T^2)$ and $(M; H'_1, H'_2; F') \# n'(\mathbb{S}^3; U_1, U_2; T^2)$ of genus h are equivalent as H-splittings. The following is known as the stabilization theorem:

Proposition 3.7 (Reidemeister [12], Singer [15]). *Arbitrary H-splittings for a 3-manifold M are stably equivalent.*

Similarly, we define, on D-splittings for a bordered 3-manifold, equivalence and stable equivalence relations. Two D-splittings $(M; H_1, H_2; F_0)$ and $(M; H'_1, H'_2; F'_0)$ for a bordered 3-manifold M are said to be *equivalent*, if there exists a homeomorphism $\psi : M \rightarrow M$ with $\psi(F_0) = F'_0$. Let $(\mathbb{D}^3; A)$ be a pair of the 3-ball \mathbb{D}^3 and a properly embedded, boundary parallel annulus A in \mathbb{D}^3 , see figure 3. The boundary ∂A divides $\partial \mathbb{D}^3$ into two disks, say, D_+ , D_- , and an annulus, say, A_0 . We choose a disk $D_0^2 \subset \partial \mathbb{D}^3$ so that $D_0^2 \cap D_+$ is a disk, $D_0^2 \cap D_-$ is a disk, and $D_0^2 \cap A_0$ is also a disk. Let $(M; H_1, H_2; F_0)$ be a D-splitting for a bordered 3-manifold M , and we choose a disk $D^2 \subset \partial M$ so that $D^2 \cap \partial H_1$ is two disks and $D^2 \cap \partial H_2$ is a disk (or $D^2 \cap \partial H_2$ is two disks and $D^2 \cap \partial H_1$ is a disk). Then, if we use these disks $D^2 \subset \partial M$ and $D_0^2 \subset \partial \mathbb{D}^3$ to form the disk-sum $M \triangle \mathbb{D}^3 \cong M$ of M and \mathbb{D}^3 , we shall obtain a new D-splitting for M with the splitting-surface $F_0 \cup A$ of genus $g(F_0) + 1$, and we denote this splitting by $(M; H_1, H_2; F_0) \triangle (\mathbb{D}^3; A)$. This process is called stabilization; it may be iterated to obtain D-splittings $(M; H_1, H_2; F_0) \triangle n(\mathbb{D}^3; A)$ of any genus $g(F_0) + n$.

It will be noticed that

- (i) $(M; H_1, H_2; F_0) \triangle (\mathbb{D}^3; A)$ depends on a disk $D^2 \subset \partial M$,

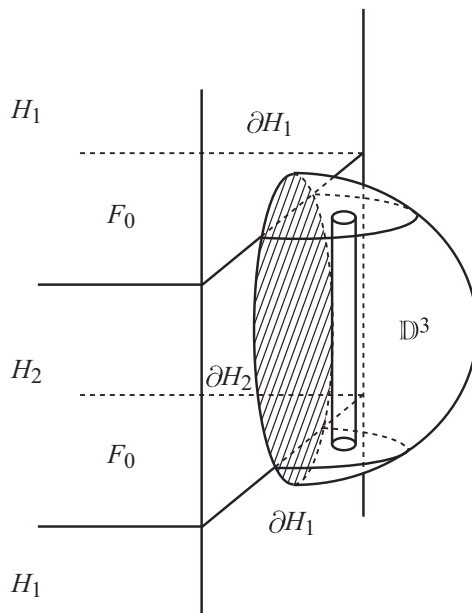


Figure 3

- (ii) $(M; H_1, H_2; F_0) \# (\mathbb{S}^3; U_1, U_2; T^2)$ is equivalent to some $(M; H_1, H_2; F_0) \triangle (\mathbb{D}^3; A)$; however $(M; H_1, H_2; F_0) \triangle (\mathbb{D}^3; A)$ is not always equivalent to any $(M; H_1, H_2; F_0) \# (\mathbb{S}^3; U_1, U_2; T^2)$.

Two D-splittings $(M; H_1, H_2; F_0)$ and $(M; H'_1, H'_2; F'_0)$ are said to be *stably equivalent*, if there exist integers n, n' with $h = g(F_0) + n = g(F'_0) + n'$ and stabilizations $(M; H_1, H_2; F_0) \triangle n(\mathbb{D}^3; A)$ and $(M; H'_1, H'_2; F'_0) \triangle n'(\mathbb{D}^3; A)$ so that these stabilizations are equivalent.

The following is a corollary to Theorem 3.4 and Proposition 3.6:

Corollary 3.8. *Arbitrary D-splittings for a bordered 3-manifold M are stably equivalent.*

4. Haken Type Theorem (1)

A 2-sphere in a 3-manifold M is *essential* if it does not bound a 3-ball in M . A 3-manifold M is *irreducible* if it contains no essential 2-sphere.

The following corresponds to the Haken Theorem 1.2.

Theorem 4.1. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M . If there exists an essential 2-sphere in M , then there exists an essential 2-sphere Σ in M such that $\Sigma \cap F_0$ consists of a single loop.*

Proof. We will give a mild generalization of this theorem in Theorem 4.3 below, and so we will not include a proof of Theorem 4.1, but simply refer the reader to Jaco's account of Haken's proof [4, chapter II] or the proof of Theorem 4.3 below. \square

Corollary 4.2. *Suppose that a bordered 3-manifold M has a decomposition*

$$M = M_1 \# \cdots \# M_u$$

as a connected sum. Then it holds that

$$\text{SDg}(M) = \text{SDg}(M_1) + \cdots + \text{SDg}(M_u).$$

Let F_0 be a compact orientable surface, and let \mathcal{J}_1 and \mathcal{J}_2 be proper 1-dimensional submanifolds in F_0 . We shall say that \mathcal{J}_1 and \mathcal{J}_2 are in *reduced position*, if $\mathcal{J}_1 \cap \mathcal{J}_2$ consists of a finite number of points in which \mathcal{J}_1 and \mathcal{J}_2 cross one another, and there is no disk on F_0 whose boundary consists of an arc in \mathcal{J}_1 and an arc in \mathcal{J}_2 .

Let M be a bordered 3-manifold and let $(M; H_1, H_2; F_0)$ be an SD-splitting for M . We call the complete systems of meridian-disks \mathcal{D}_1 of H_1 and \mathcal{D}_2 of H_2 which satisfy the condition (vi*) a *special complete systems of meridian-disks*. These special complete systems of meridian-disks \mathcal{D}_1 of H_1 and \mathcal{D}_2 of H_2 are said to be *irreducible* if $\mathcal{J}_1 = \mathcal{D}_1 \cap F_0$ and $\mathcal{J}_2 = \mathcal{D}_2 \cap F_0$ are in reduced position in F_0 .

Theorem 4.3. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M , and let $\mathcal{D}_j = \{D_{j1}, \dots, D_{jm}\}$ be a special complete system of meridian-disks of H_j ($j = 1, 2$), and we suppose that \mathcal{D}_1 and \mathcal{D}_2 are irreducible. Let Σ be a disjoint union of essential 2-spheres in M . Then there exist a disjoint union of essential 2-spheres Σ^* and a complete system of meridian-disks \mathcal{D}_2^* of H_2 such that*

- (i) Σ^* is obtained from Σ by ambient 1-surgery and isotopy,
- (ii) each component of Σ^* meets F_0 in a single loop,
- (iii) $\mathcal{D}_1 \cap \Sigma^* = \emptyset$, $\mathcal{D}_2^* \cap \Sigma^* = \emptyset$, and $\mathcal{D}_2^* \cap (F_{j1} \cup \cdots \cup F_{jm}) = \mathcal{D}_2 \cap (F_{j1} \cup \cdots \cup F_{jm})$, where F_{ji} is the planar surface $\partial H_j \cap B_i$, B_i a connected component of ∂M .

Proof. We choose a 1-dimensional spine S_{2i} of the planar surface F_{2i} so that S_{2i} consists of simple loops based at the point x_i and each loop intersected with D_2 at a single point ($i = 1, 2, \dots, m$). Then we can choose a 1-dimensional spine S_2 of H_2 so that $S_2 \cap D_{2i}$ consists of a single point ($i = 1, 2, \dots, m$) and $S_2 \cap \partial H_2 = S_{21} \cup \cdots \cup S_{2m}$. We may suppose that S_2 intersects transversally with Σ at a finite number of points. Since H_2 is a regular neighborhood of S_2 , we may assume that Σ intersects with H_2 at a finite number of disks, say, $\sigma_1, \dots, \sigma_n$.

Let $\Sigma_0 = \text{Cl}(\Sigma - (\sigma_1 \cup \cdots \cup \sigma_n); \Sigma)$. Then $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g})$ consists of a finite number of simple loops and proper arcs. Since H_1 is irreducible, we can remove all simple loops by cut-and-paste, and so we may assume that $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g})$ consists of a finite number of proper arcs, say, $\alpha_1, \dots, \alpha_k$. Since $\Sigma_0 \cap F_{1i} = \emptyset$ for $i = 1, 2, \dots, m$, we can choose an innermost arc, say, α_1 , on one of D_{11}, \dots, D_{1g} , say, D_{11} , if $\Sigma_0 \cap (D_{11} \cup \cdots \cup D_{1g}) \neq \emptyset$. Let $\Delta \subset D_{11}$ be the disk cut off by α_1 so that

$$\Delta \cap \Sigma_0 = \partial \Delta \cap \Sigma_0 = \alpha_1, \quad \Delta \cap (F_{11} \cup \cdots \cup F_{1m}) = \emptyset.$$

Now, we may apply the same argument as that of Jaco [6, II7–II9]; that is, we can deform Σ along Δ (by isotopy of type A) so that the new Σ^* does not meet at α_1 . By the repetition of the procedure, we can get rid of all intersections $\alpha_1, \dots, \alpha_k$ of $\Sigma^* \cap \mathcal{D}_1$. Now, it is easy to see that the new Σ^* satisfies the conditions (i) and (ii), and the condition $\mathcal{D}_1 \cap \Sigma^* = \emptyset$ from (iii).

Since $H_2 \cap \Sigma^*$ consists of a finite number of disks and $\Sigma^* \cap (F_{21} \cup \cdots \cup F_{2m}) = \emptyset$, we can choose, if necessary, a complete system of meridian-disks \mathcal{D}_2^* of H_2 so that \mathcal{D}_2^* satisfies the other conditions in (iii), completing the proof. \square

5. Haken type theorem (2)

A proper disk in a bordered 3-manifold M is said to be *essential* if it does not cut off a 3-ball from M . Using essential disks, Gross[3] and Swarup [16] have formulated another prime decomposition theorem under disk-sum (i.e., boundary connected sum) for a bordered 3-manifold.

Now the following question immediately comes to mind:

Question and Example 5.1. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M . If there exists an essential proper disk in M , then does there exist an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc?*

The answer is NO in general. The following counterexample is due to Dr. Kanji Morimoto. Let K be a simple loop on the boundary $S^1 \times S^1$ of the solid torus $D^2 \times S^1$ such that $K \cap D^2 = K \cap \partial D^2$ consists of two crossing points, where D^2 is a standard meridian-disk of $D^2 \times S^1$. Let $J \subset D^2$ be a simple proper arc joining the two points. Let $H_1 = N(K \cup J; D^2 \times S^1)$, and $H_2 = \text{Cl}(D^2 \times S^1 - H_1; D^2 \times S^1)$. Then we have an SD-splitting $(D^2 \times S^1; H_1, H_2; F_0)$ for $D^2 \times S^1$ of genus 2, where F_0 is the surface $\text{Cl}(\partial H_1 \cap \text{Int}(D^2 \times S^1); D^2 \times S^1)$. The meridian-disk D^2 is an essential proper disk in $D^2 \times S^1$ which is unique up to ambient isotopy of $D^2 \times S^1$, and $D^2 \cap F_0$ consists of two arcs. It will be noticed that $D^2 \times S^1$ has an SD-splitting of genus 1, and the above splitting is of genus 2. \square

Proposition 5.2. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for a bordered 3-manifold M . If there exists an essential 2-sphere in M which is not boundary parallel, then there exists an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc.*

Proof. By Theorem 4.1 (or 4.3), we have an essential 2-sphere Σ in M such that $\Sigma \cap F_0$ consists of a single loop. Using this Σ , we can easily obtain a required essential disk Δ . \square

The following lemma corresponds to Theorem 4.3.

Lemma 5.3. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for an irreducible bordered 3-manifold M . If there exists an essential proper disk in M , then there exist an essential proper disk Δ in M and a special complete system of meridian-disks $\mathcal{D}_j = \{D_{j1}, \dots, D_{jg}\}$ of H_j ($j = 1, 2$) satisfying the following:*

- (i) $\Delta \cap F_0$ consists of a finite number of proper arcs,
- (ii) $\Delta \cap H_j$ consists of a finite number of proper disks, and each component is essential in H_j ($j = 1, 2$), and
- (iii) $\Delta \cap D_2 = \emptyset$.

Proof. We choose a 1-dimensional spine S_2 of H_2 in the same way as that of the proof of Theorem 4.3. Then, we may consider H_2 as a regular neighborhood of S_2 .

Let \square be an essential proper disk in M . We may assume that \square intersects with S_2 transversally in a finite number of points, and so $\square \cap H_2$ consists of a finite number of proper disks, which are regular neighborhoods of $\square \cap S_2$ in \square . Now $\square \cap F_0$ consists of a finite number of proper arcs and loops. We can remove the loops in the same way as in the proof of Theorem 4.3 (see Jaco [6]), and let Δ be the new disk. It is easy to see that Δ satisfies the conditions (i) and (ii). If we cut H_2 along Δ then we have some handlebodies, and so we can choose a complete system of meridian-disks \mathcal{D}_2 of H_2 with the condition (iii). This completes the proof. \square

Using this Lemma, we can prove the following:

Proposition 5.4. *Let $(M; H_1, H_2; F_0)$ be an SD-splitting for an irreducible bordered 3-manifold M with connected boundary B of genus g . If there exists an essential proper disk in M and $\text{SDg}(M) = g$, then there exists an essential proper disk Δ in M such that $\Delta \cap F_0$ consists of a single arc.*

Proof. Let $\Delta \subset M$ be an essential proper disk, and \mathcal{D}_j be a special complete system of meridian-disks of H_j ($j = 1, 2$) that satisfy the conditions of Lemma 5.3. We cut H_j along \mathcal{D}_j ; we have a 3-ball D_j^3 . On the boundary ∂D_j^3 , F_{j1} appears as a disk from the condition (vi*)-(b). Using Δ we construct the required disk by the condition (iii). The proof is not so hard but fairly complicated, and we omit it here. \square

As a corollary to this Proposition, we have the following characterization of handlebodies by SD-splittings.

Corollary 5.5. *Let M be an irreducible bordered 3-manifold with connected boundary B of genus g , and we suppose that M contains an essential proper disk. Then it holds that*

$$\text{SDg}(M) = g \iff M \text{ is a handlebody of genus } g.$$

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